

On Harrap's conjecture in Diophantine approximation

by Nikolay Moshchevitin¹

Abstract. We prove a conjecture due to Stephen Harrap on inhomogeneous linear Diophantine approximation related to $\text{BAD}(\alpha, \beta)$ sets.

1. The result.

Let $\alpha, \beta \in (0, 1)$, $\alpha + \beta = 1$. Let $\|\cdot\|$ denote the distance to the nearest integer. Consider the set

$$\text{BAD}(\alpha, \beta) = \{(\theta_1, \theta_2) \in \mathbb{R}^2 : \inf_{q \in \mathbb{Z}_+} \max(q^\alpha \|q\theta_1\|, q^\beta \|q\theta_2\|) > 0\}.$$

For $\Theta = (\theta_1, \theta_2) \in \mathbb{R}^2$ consider the set

$$\text{BAD}^\Theta(\alpha, \beta) = \{(\eta_1, \eta_2) \in \mathbb{R}^2 : \inf_{q \in \mathbb{Z}_+} \max(q^\alpha \|q\theta_1 - \eta_1\|, q^\beta \|q\theta_2 - \eta_1\|) > 0\}.$$

In [2] it was proved that if $\Theta \in \text{BAD}(\alpha, \beta)$ then the set $\text{BAD}^\Theta(\alpha, \beta)$ has full Hausdorff dimension in \mathbb{R}^2 . Moreover it was noted there that in this case it is possible to establish the winning property of the set $\text{BAD}^\Theta(\alpha, \beta)$.

It was conjectured in [2] that the set $\text{BAD}^\Theta(\alpha, \beta)$ should be a set of full Hausdorff dimension without the additional assumption $\Theta \in \text{BAD}(\alpha, \beta)$. In the case $\alpha = \beta = 1/2$ this is true (it follows from Khintchine's and Jarnik's approach [3, 4]). In the present note we give a solution to the problem from [2]. For simplicity reason we restrict ourselves by the case $\alpha = 2/3, \beta = 1/3$.

Theorem. *For any Θ the set $\text{BAD}^\Theta(2/3, 1/3)$ is non-empty. Moreover it has full Hausdorff dimension.*

2. Best approximations.

We may suppose that $1, \theta_1, \theta_2$ are linearly independent over \mathbb{Z} . Otherwise the result is obvious.

We define $\mathbf{m} = (m_1, m_2) \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$ to be a best approximation vector if in the parallelepiped

$$\{(x_0, x_1, x_2) \in \mathbb{R}^3 : \max(|x_1|, |x_2|^2) \leq \max(|m_1|, |m_2|^2), |x_0 + \theta_1 x_1 + \theta_2 x_2| \leq \|\theta_1 m_1 + \theta_2 m_2\|\}$$

there is no integer points different from the points $(0, 0, 0), \pm(m_0, m_1, m_2)$, where $m_0 \in \mathbb{Z}$ is defined from $\|\theta_1 m_1 + \theta_2 m_2\| = |m_0 + \theta_1 m_1 + \theta_2 m_2|$. All the best approximation vectors should be arranged in the infinite sequence

$$\mathbf{m}_\nu = (m_{\nu,1}, m_{\nu,2}), \quad \nu = 1, 2, 3, \dots$$

in such a way that the values

$$M_\nu = \sqrt{\max(|m_{\nu,1}|, |m_{\nu,2}|^2)}$$

form an increasing sequence ($M_{\nu+1} > M_\nu$), and the values

$$\zeta_\nu = \|\theta_1 m_{1,\nu} + \theta_2 m_{2,\nu}\|$$

form a decreasing sequence ($\zeta_{\nu+1} < \zeta_\nu$). From the Minkowski convex body theorem it follows that

$$\zeta_\nu M_{\nu+1}^3 \leq 1. \tag{1}$$

¹research is supported by RFBR grant No.12-01-00681-a and by the grant of Russian Government, project 11. G34.31.0053.

Note that if \mathbf{m}_ν is a best approximation vector then in any parallelepiped of the form

$$\{(x_0, x_1, x_2) \in \mathbb{R}^3 : \max(2|x_1 - \xi_1|, 4|x_2 - \xi_2|^2) \leq M_\nu^2, |x_0 + \theta_1 x_1 + \theta_2 x_2 - \xi_0| \leq \frac{\zeta_\nu}{2}\} \quad (2)$$

with $(\xi_0, \xi_1, \xi_2) \in \mathbb{R}^3$ there exists not more than one integer point.

As the set

$$\begin{aligned} & \{(x_0, x_1, x_2) \in \mathbb{R}^3 : \max(|x_1|, |x_2|^2) \leq (2M_\nu)^2, |x_0 + \theta_1 x_1 + \theta_2 x_2| \leq \zeta_\nu\} \setminus \\ & \{(x_0, x_1, x_2) \in \mathbb{R}^3 : \max(|x_1|, |x_2|^2) \leq M_\nu^2, |x_0 + \theta_1 x_1 + \theta_2 x_2| \leq \zeta_\nu\} \end{aligned}$$

may be partitioned into 2×28 sets of the form (2) and the lattice \mathbb{Z}^3 is $\mathbf{0}$ -symmetric we see that

$$M_{\nu+28} \geq 2M_\nu \quad (3)$$

for every value of ν .

There are two kinds of best approximation vectors. For some best approximation vectors we have

$$M_\nu = \sqrt{\max(|m_{\nu,1}|, |m_{\mu,2}|^2)} = \sqrt{|m_{\nu,1}|}. \quad (4)$$

For other best approximation vectors we have

$$M_\nu = \sqrt{\max(|m_{\nu,1}|, |m_{\mu,2}|^2)} = |m_{\nu,2}|. \quad (5)$$

All the best approximation vectors satisfying (4) form the sequence

$$\mathbf{m}_\nu^{[1]} = (m_{\nu,1}^{[1]}, m_{\nu,2}^{[1]}), \quad \nu = 1, 2, 3, \dots$$

in such a way that the values

$$M_\nu^{[1]} = \sqrt{\max(|m_{\nu,1}^{[1]}|, |m_{\mu,2}^{[1]}|^2)} = \sqrt{|m_{\nu,1}^{[1]}|}$$

form an increasing sequence. All the best approximation vectors satisfying (5) form the sequence

$$\mathbf{m}_\nu^{[2]} = (m_{\nu,1}^{[2]}, m_{\nu,2}^{[2]}), \quad \nu = 1, 2, 3, \dots$$

in such a way that the values

$$M_\nu^{[2]} = \sqrt{\max(|m_{\nu,1}^{[2]}|, |m_{\mu,2}^{[2]}|^2)} = |m_{\nu,2}^{[2]}|$$

form an increasing sequence. From (3) we see that

$$M_{\nu+28}^{[1]} \geq 2M_\nu^{[1]}, \quad M_{\nu+28}^{[2]} \geq 2M_\nu^{[2]}. \quad (6)$$

3. Linear form bounded from zero.

In this section we prove that there exists a vector $\eta = (\eta_1, \eta_2)$ satisfying

$$\inf_\nu ||\eta_1 m_{\nu,1} + \eta_2 m_{\nu,2}|| > 0. \quad (7)$$

Standard transference argument (see [1], Chapter V) in view of (1) shows that for such η we have $\eta \in \text{BAD}^\Theta(2/3, 1/3)$. Moreover standard argument with resonance sets (see [2, 5]) shows that the set of η satisfying (7) is a set of full Hausdorff dimension.

To get (7) it is enough to show that

$$\inf_{\nu} ||\eta_1 m_{\nu,1}^{[j]} + \eta_2 m_{\nu,2}^{[j]}|| > 0, \quad j = 1, 2. \quad (8)$$

Let R be a large positive integer. Put

$$\delta = \frac{1}{R^3}. \quad (9)$$

We will describe the inductive process. Its base is trivial. Suppose that for a positive integer n we have a rectangle $B \subset \mathbb{R}^2$ of a form

$$B = \left[b_1, b_1 + \frac{\delta}{R^{2n}} \right] \times \left[b_2, b_2 + \frac{\delta}{R^n} \right].$$

We suppose that for all $\eta \in B$ and for all best approximation vectors

$$\mathbf{m}_{\nu}^{[1]}, \quad M_{\nu}^{[1]} \leq R^n$$

and for all best approximation vectors

$$\mathbf{m}_{\nu}^{[2]}, \quad M_{\nu}^{[2]} \leq R^n$$

we have

$$||\eta_1 m_{\nu,1}^{[j]} + \eta_2 m_{\nu,2}^{[j]}|| > \varepsilon \quad \text{with} \quad \varepsilon = \frac{\delta}{R}. \quad (10)$$

By dividing the segments $[b_1, b_1 + \frac{\delta}{R^{2n}}]$, $[b_2, b_2 + \frac{\delta}{R^n}]$ into R^2 and R equal parts correspondingly we get a partition of B into R^3 smaller rectangles B' of the form

$$B' = \left[b'_1, b'_1 + \frac{\delta}{R^{2(n+1)}} \right] \times \left[b'_2, b'_2 + \frac{\delta}{R^{n+1}} \right]. \quad (11)$$

We should show that there exist many subrectangles of the form (11) such that for all the points from these rectangles we have (10) for all the best approximation vectors

$$\mathbf{m}_{\nu}^{[1]}, \quad R^n < M_{\nu}^{[1]} \leq R^{n+1}, \quad (12)$$

$$\mathbf{m}_{\nu}^{[2]}, \quad R^n < M_{\nu}^{[2]} \leq R^{n+1}. \quad (13)$$

That will be enough.

1. We will deal with the best approximation vectors (12). For a single vector $\mathbf{m}_{\nu}^{[1]}$ from (12) consider the collection of parallel lines

$$\mathcal{L}_{\nu}^{[1]} = \bigcup_{c \in \mathbb{Z}} l_{\nu}^{[1]}(c), \quad l_{\nu}^{[1]}(c) = \{(x_1, x_2) \in \mathbb{R}^2 : m_{\nu,1}^{[1]}x_1 + m_{\nu,2}^{[1]}x_2 = c\}.$$

Consider a rectangle $B' \subset B$ of the form (11) which has a point (ξ_1, ξ_2) with $|\xi_1 m_{\nu,1}^{[1]} + \xi_2 m_{\nu,2}^{[1]} - c| < \varepsilon$, with some $c \in \mathbb{Z}$. Let $H_{\nu}^{[1]}$ be the number of such subrectangles.

Fix x_2 , and consider the one-dimensional section with x_2 fixed. Then the point $\left(\frac{-m_{\nu,2}^{[1]}x_2 + c}{m_{\nu,1}^{[1]}}, x_2 \right)$ belongs to the line $l_{\nu}^{[1]}(c)$. The section of the subrectangle B' under the consideration must completely lie in the segment $J(c)$ of the form

$$\left[\left(\frac{-m_{\nu,2}^{[1]}x_2 + c}{m_{\nu,1}^{[1]}} - \frac{\varepsilon}{|m_{\nu,1}^{[1]}|} - \frac{\delta}{R^{2(n+1)}} - \frac{k^{[1]}\delta}{R^{n+1}}, x_2 \right), \left(\frac{-m_{\nu,2}^{[1]}x_2 + c}{m_{\nu,1}^{[1]}} + \frac{\varepsilon}{|m_{\nu,1}^{[1]}|} + \frac{\delta}{R^{2(n+1)}} + \frac{k^{[1]}\delta}{R^{n+1}}, x_2 \right) \right],$$

where

$$k^{[1]} = \frac{|m_{\nu,2}^{[1]}|}{|m_{\nu,1}^{[1]}|} \leq \frac{1}{\sqrt{|m_{\nu,1}^{[1]}|}} \leq \frac{1}{R^n}$$

(we use (4) and the lower bound from (12)). For the length $\Delta_\nu^{[1]}$ of such a segment $J(c)$ we have the upper bound

$$\Delta_\nu^{[1]} \leq \frac{2\varepsilon + 2\delta R^{-2} + 2\delta R^{-1}}{R^{2n}}.$$

We suppose R to be large enough, so by (9) we have

$$\frac{1}{|m_{\nu,1}^{[1]}|} - \Delta_\nu^{[1]} > \frac{\delta}{R^{2n}}.$$

We see that each section of rectangle B with fixed value of x_2 can intersect not more than just one segment $J(c)$. Now

$$H_\nu^{[1]} \leq \frac{\Delta_\nu^{[1]} \times \delta / R^n}{\delta^2 / R^{3(n+1)}} \leq \frac{2\varepsilon}{\delta} R^3 + 3R^2 = 5R^2.$$

The number of vectors $\mathbf{m}_\nu^{[1]}$ satisfying (12) is $\leq 56 \log_2 R$ due to (6).

We come to the following conclusion. *The total number of “dangerous” subrectangles $B' \subset B$ which are “killed” by the lines corresponding to the vectors (12) is less than*

$$\sum_{\nu \text{ satisfy (12)}} H_\nu^{[1]} \leq 600 R^2 \log_2 R.$$

2. We will deal with the best approximation vectors (13). For a single vector $\mathbf{m}_\nu^{[2]}$ from (13) consider the collection of parallel lines

$$\mathcal{L}_\nu^{[2]} = \bigcup_{c \in \mathbb{Z}} l_\nu^{[2]}(c), \quad l_\nu^{[2]}(c) = \{(x_1, x_2) \in \mathbb{R}^2 : m_{\nu,1}^{[2]}x_1 + m_{\nu,2}^{[2]}x_2 = c\}.$$

Consider a rectangle $B' \subset B$ of the form (11) which has a point (ξ_1, ξ_2) with $|\xi_1 m_{\nu,1}^{[2]} + \xi_2 m_{\nu,2}^{[2]} - c| < \varepsilon$, with some $c \in \mathbb{Z}$. Let $H_\nu^{[2]}$ be the number of such subrectangles.

Fix x_1 , and consider the one-dimensional section with x_1 fixed. Then the point $\left(x_1, \frac{-m_{\nu,1}^{[2]}x_1 + c}{m_{\nu,2}^{[2]}}\right)$ belongs to the line $l_\nu^{[2]}(c)$. The section of the subrectangle B' under the consideration must completely lie in the segment

$$\left[\left(x_1, \frac{-m_{\nu,1}^{[2]}x_1 + c}{m_{\nu,2}^{[2]}} - \frac{\varepsilon}{|m_{\nu,2}^{[2]}|} - \frac{\delta}{R^{(n+1)}} - \frac{k^{[2]}\delta}{R^{2(n+1)}} \right), \left(x_1, \frac{-m_{\nu,1}^{[2]}x_1 + c}{m_{\nu,2}^{[2]}} + \frac{\varepsilon}{|m_{\nu,2}^{[2]}|} + \frac{\delta}{R^{n+1}} + \frac{k^{[2]}\delta}{R^{2(n+1)}} \right) \right],$$

where

$$k^{[2]} = \frac{|m_{\nu,1}^{[2]}|}{|m_{\nu,2}^{[2]}|} \leq |m_{\nu,2}^{[2]}| \leq R^{n+1}$$

(we use (5) and the upper bound from (13)). For the length $\Delta_\nu^{[2]}$ of such a segment we have the upper bound

$$\Delta_\nu^{[2]} \leq \frac{2\varepsilon + 4\delta R^{-1}}{R^n}.$$

So by (9) we have

$$\frac{1}{|m_{\nu,2}^{[2]}|} - \Delta_{\nu}^{[2]} > \frac{\delta}{R^n}.$$

Now

$$H_{\nu}^{[2]} \leq \frac{\Delta_{\nu}^{[2]} \times \delta / R^{2n}}{\delta^2 / R^{3(n+1)}} \leq \frac{2\varepsilon}{\delta} R^3 + 4R^2 = 6R^2.$$

The number of vectors $\mathbf{m}_{\nu}^{[2]}$ satisfying (13) is $\leq 28 \log_2 R$ due to (6).

We come to the following conclusion. *The total number of “dangerous” subrectangles $B' \subset B$ which are “killed” by the lines corresponding to the vectors (13) is less than*

$$\sum_{\nu \text{ satisfy (13)}} H_{\nu}^{[2]} \leq 400 R^2 \log_2 R.$$

Combining together the conclusions from **1.** and **2.** we see that *the total number of “bad” subrectangles (11) is less than*

$$1000 R^2 \log_2 R.$$

The whole number of subrectangles is R^3 . So there exist at $R^3 - 1000 R^2 \log_2 R$ subrectangles B' for which the desired property is satisfied for the $(n+1)$ -th step. It is enough to prove the existence of η . The full Hausdorff dimension of the set of such η 's follows from the analysis of the resonance sets $\mathcal{L}_{\nu}^{[j]}$.

References

- [1] J.W.S. Cassels, An introduction to Diophantine approximations, Cambridge Univ. Press, 1957.
- [2] S. Harrap, Twisted inhomogeneous Diophantine approximation and badly approximable sets, Acta Arithmetica, 151 (2012), 55 - 82; preprint available at arXiv:1003.2362v4.
- [3] A. Khintchine, Über die angenäherte Auflösung Linearer Gleichungen in ganzen Zahlen. // Acta Arithmetica, 2 (1936), 161 - 172.
- [4] V. Jarník, On lineárních nehomogenních diofantických aproximacích, Rozpravy II. Třidy České Akademie, Ročník LI, Číslo 29, 1 - 21 (1941).
- [5] S. Kristensen, R. Thorn, S. Velani, Diophantine approximation and badly approximable sets, Adv. Math. 203 (2006), no.1, 132- 169.